MATH 3060 Assignment 7 solution

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1. (a) A subset is nowhere dense if and only if the complement of its closure is dense. So it suffices to show $\bar{\bar{E}} = \bar{E}$. In other words, it suffices to show $\bar{E}$ is closed. In fact, let $x \notin \bar{E}$, by definition, there exists a small ball $B_r(x)$ of x so that $B_r(x) \subset E^c$. But then for each $x' \in B_r(x)$, we can find another small ball $B_{r'}(x')$ so that $B_{r'}(x') \subset B_r(x) \subset E^c$. This shows that $B_r(x) \subset \bar{E}^c$ and thus $\bar{E}$ is closed.

   (b) Let S be a subset of E, then $\bar{S} \subset \bar{E}$, and thus $\bar{S}^c \supset \bar{E}^c$ is dense.

2. (a) Let $f \in C[0,1]$, and $r > 0$. We need to show that $B_r(f)$ contains some f' with $\int_0^1 f' \neq 0$. If $\int_0^1 f \neq 0$, we can take $f' = f$. If $\int_0^1 f = 0$, then we can take $f' = f + \frac{r}{2}$.

   (b) First note that the set

   $$S = \{f \in C[0,1] : f(0.1) = 2\}$$

   is closed. In fact, if $f_n \in S$, and $f_n \to f$ uniformly, then $f(0.1) = \lim f_n(0.1) = 2$. So $f \in S$.
   Next suppose $g \notin S$, and $r > 0$, we need to find a $g' \in B_r(g) \setminus S$. If $g \notin S$, then we can take $g' = g$. If $g \in S$, then take $g' = g + \frac{r}{2}$, we have $g'(0.1) = 2 + \frac{r}{2} \neq 2$, so $g' \notin S$.

3. From homework 4, question 3b) we know that H is closed.
   Now suppose $\{x_m\} \notin H$, and $r > 0$, we need to find $\{x'_m\} \in B_r(\{x_m\}) \setminus H$. If $\{x_m\} \notin H$, then we can take $\{x'_m\} = \{x_m\}$. If $\{x_m\} \in H$, then we can find $N \in \mathbb{N}$ so that $\frac{r}{2} > \frac{2}{N}$. We can define $x'_n = x_n$ if $n \neq N$ and $x'_N = x_N + \frac{r}{2}$. Then $\{x'_m\} \in B_r(\{x_m\})$ and

   $$x'_N > x_n + \frac{2}{N} \geq \frac{1}{N}.$$

4. Let U be an open set, then $\partial U = \bar{U} \cap U^c$ is the intersection of two closed subsets, and thus closed. Now let $x \notin \partial U$, and $r > 0$, we need to find $x' \in B_r(x)$ so that $x' \notin \partial U$. If $x \notin \partial U$, then we can take $x' = x$. If $x \in \partial U$, then we can take $x' \in U$ by the definition of boundary. Then $x' \notin \partial U$ because $\partial U \subset U^c$.

Conversely, suppose F is nowhere dense and closed, then take $U = F^c$. Then U must be open and dense. Now we can conclude $\partial U = F \cap \bar{U} = F$ because the closure of U is the whole space.